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On the elastic-plastic torsion problem

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SUMMARY

It is shown that solving the elastic-plastic torsion problem is equivalent to solving a nonlinear boundary value problem. The equivalence is valid for a strictly convex cross-section and angles of twist large enough that the plastic region encloses the elastic region. The advantage of this formulation is that no free boundaries are present.

1. Introduction

In the torsion problem for a prismatic elastic-plastic bar, a stress function $\Psi(x, y)$ is sought satisfying

$$\nabla^2 \Psi = -2G\theta, \tag{1}$$

when $|\nabla \Psi|^2 < k^2$. Otherwise,

$$|\nabla \Psi|^2 = k^2. \tag{2}$$

$|\nabla \Psi|^2 > k^2$ is not allowed. In (1) and (2), G is the elastic shear modulus, k is the yield stress in shear, and θ is the angle of twist per unit length of the bar. An unknown elastic-plastic boundary separates the regions in which (1) and (2) apply. Across this boundary, Ψ and its first derivatives are continuous. Finally, for a simply-connected cross-section, $\Psi = 0$ on the cross-section boundary. For a detailed discussion of this problem, see [1].

The essential difficulty in this problem is that the regions in which (1) and (2) hold are not known in advance. Thus, this problem is analytically difficult, in spite of the fact that solving either (1) or (2) in any given known region is not. Only two solutions appear to be known explicitly: the solution for a circular cross-section, and a solution found by Sokolovsky for a certain one-parameter family of cross-sections. Both are discussed in [1].

In this paper, we show that solving the elastic-plastic torsion problem is equivalent to solving a nonlinear boundary value problem. The equivalence is valid when

(a) the cross-section is strictly convex

(b) the elastic region lies entirely inside the plastic region.

Restriction (b) also applies to Sokolovsky's solution. The advantage of this formulation is that the original problem involving a free boundary is replaced by a problem involving known boundaries only.

The equivalence of the problems is developed in Section 2. In Section 3, Sokolovsky's solution is rederived from this point of view. In what follows the results in [1], particularly those concerning the solution of (2), are used without proof.

2. Analysis

Let the cross-section boundary be described parametrically by the equations

$$x = x_0(\phi), \quad y = y_0(\phi). \tag{3}$$

The parameter ϕ is the angle between the x-axis and the normal to the curve (3) at the point $x = x_0(\phi), y = y_0(\phi)$; see Figure 1. Analytically,

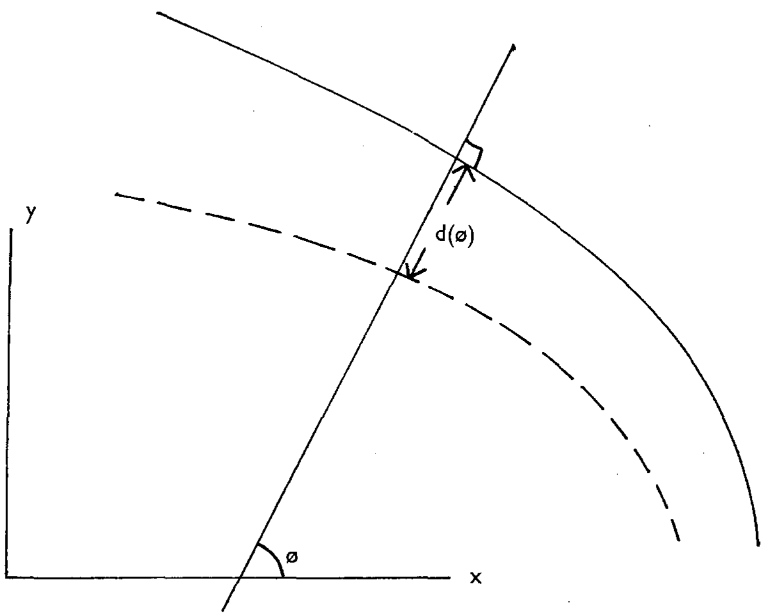

Figure 1.

$$\tan \phi = -x_0'(\phi)/y_0'(\phi). \tag{4}$$

The curve (3) admits this parametrization only if it encloses a convex region and does not contain any straight line segments; otherwise, values of ϕ are not in one to one correspondence with the boundary points. If the parametrization (3), (4) is not possible, this method of solution does not apply.

As in Figure 1, denote by $d(\phi)$ the distance between the point $x = x_0(\phi), y = y_0(\phi)$ and the elastic-plastic boundary measured along the normal to the cross-section boundary. Then the elastic-plastic boundary has the parametric form

$$x = x_0(\phi) - d(\phi)\cos\phi, \quad y = y_0(\phi) - d(\phi)\sin\phi.$$

The elastic region is assumed to be enclosed entirely by the plastic region. Therefore, the values of the elastic stress function $\Psi(x, y)$ and its derivatives are known on the elastic-plastic boundary. In view of the definition of ϕ and the solution of equation (2),

$$\Psi(\phi) = +k\,d(\phi), \quad \frac{\partial \Psi}{\partial x}(\phi) = -k\cos\phi, \quad \frac{\partial \Psi}{\partial y}(\phi) = -k\sin\phi. \tag{5}$$

Define

$$u(x, y) = \Psi(x, y)/k.$$

Note that u has the dimensions of length. Let $p = \partial u/\partial x$, $q = \partial u/\partial y$. Then (5) becomes

$$u(\phi) = +d(\phi), \quad p(\phi) = -\cos\phi, \quad q(\phi) = -\sin\phi. \tag{6}$$

Now apply the Legendre transformation

$$X = p, \quad Y = q, \quad U = xp + yq - u, \quad P = x, \quad Q = y.$$

Since the plastic stress function satisfies (2), the elastic-plastic boundary is transformed into the unit circle $X^2 + Y^2 = 1$. In terms of polar coordinates R, Θ in the X–Y plane, $\tan\Theta = Y/X = q/p$. Therefore, in view of (6), on the unit circle,

$$\tan\Theta = \tan\phi. \tag{7}$$

The elastic equation (1) is transformed to

$$\nabla^2 U = -2\theta G\left[\frac{\partial^2 U}{\partial X^2}\frac{\partial^2 U}{\partial Y^2} - \left(\frac{\partial^2 U}{\partial X\partial Y}\right)^2\right] \Big/ k. \tag{8}$$

Because of (7), the boundary conditions on (8) are

$$\begin{aligned} -U(\Theta) &= \cos\Theta[x_0(\Theta) - d(\Theta)\cos\Theta] + \sin\Theta[y_0(\Theta) - d(\Theta)\sin\Theta] + d(\Theta) \\ &= \cos\Theta\, x_0(\Theta) + \sin\Theta\, y_0(\Theta). \end{aligned} \tag{9}$$

Thus, the boundary conditions only involve the known functions x_0 and y_0.

Of course, the function $u(x, y)$ is unknown. Suppose, however, that the solution of (8) and (9) is known. Then u can be found by inverting the Legendre transformation: solve

$$x = P(X, Y), \quad y = Q(X, Y) \tag{10}$$

for X and Y and then substitute into $u = XP + YQ - U$.

To complete the solution, the elastic-plastic boundary is required. In the X–Y plane, the elastic-plastic boundary is the unit circle $X^2 + Y^2 = 1$; to find its equations in the x–y plane, use (7) and the formulas $P = x$, $Q = y$:

$$\begin{aligned} x(\Theta) &= P(\Theta) = \frac{\partial U}{\partial X}(\cos\Theta, \sin\Theta), \\ y(\Theta) &= Q(\Theta) = \frac{\partial U}{\partial Y}(\cos\Theta, \sin\Theta). \end{aligned} \tag{11}$$

Both functions on the right sides in (11) are known from the solution of (8) and (9).

Thus, solution of the original free boundary problem is replaced by the solution of equation (8) subject to boundary conditions (9) on the unit circle.

This formulation suggests an inverse method for the elastic-plastic torsion problem. Suppose that solutions of (8) are known which have boundary values independent of θ on the unit circle.

Then a possible cross-section boundary could be found from (9). So far, however, we are unable to find any such solution of (8) other than one leading to Sokolovsky's solution.

A number of free boundary problems have been treated by transforming the unknown boundary into a known boundary. One example is the elastic-plastic antiplane shear problem; see [2] and the references there. For the problem of water seepage through the ground, see [3]. The idea seems to be due originally to Kirchhoff, who applied it in the free streamline problem in the theory of potential flow [4].

3. Sokolovsky's solution

The cross-sectional boundary in parametric form [1] is

$$x = (b + 3c)\cos\phi - c\cos^3\phi, \quad y = b\sin\phi + c\sin^3\phi.$$

Therefore, in the X–Y plane, (8) must be solved subject to the boundary condition

$$\begin{aligned} -U(\cos\Theta, \sin\Theta) &= (b + 3c)\cos^2\Theta - c\cos^4\Theta + b\sin^2\Theta + c\sin^4\Theta \\ &= (b + c) + c\cos^2\Theta \end{aligned} \tag{12}$$

when $R = 1$. Assume a solution of (8) of the form

$$U(X, Y) = \alpha X^2 + \beta Y^2 + \gamma. \tag{13}$$

Then substituting into (8),

$$\alpha + \beta = -4(G/k)\theta\alpha\beta. \tag{14}$$

Equation (13) satisfies the boundary condition (12) if

$$-[(\gamma + \beta) + (\alpha - \beta)\cos^2\Theta] = (b + c) + c\cos^2\Theta.$$

Therefore,

$$\gamma + \beta = -(b + c), \quad -\alpha + \beta = c. \tag{15}$$

Solving (14) and (15),

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = -\frac{k}{4G\theta} \mp \frac{c}{2} - \sqrt{\frac{k^2}{16G^2\theta^2} + \frac{c^2}{4}}. \tag{16}$$

The function $u(x, y)$ is found by solving (10). Substituting (13) into (10),

$$x = P = 2\alpha X, \quad y = Q = 2\beta Y.$$

Therefore

$$u(x, y) = XP + YQ - U = \frac{1}{4\alpha}x^2 + \frac{1}{4\beta}y^2 - \gamma. \tag{17}$$

The elastic-plastic boundary is found from (11). It is the ellipse

$$x(\Theta) = 2\alpha\cos\Theta, \quad y(\Theta) = 2\beta\sin\Theta. \tag{18}$$

Note that the elastic stress function defined by (16) and (17) and the elastic-plastic boundary (18) agree with the formulas of [1] p. 74.

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